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Tunnelling out of a time-dependent well

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Received 25 April 2005, in final form 17 May 2005

Published 15 June 2005

Online at stacks.iop.org/JPhysA/38/5993

Abstract

Solutions to explicit time-dependent problems in quantum mechanics are rare. In fact, all known solutions are coupled to specific properties of the Hamiltonian and may be divided into two categories: one class consists of time-dependent Hamiltonians which are not higher than quadratic in the position operator, like e.g. the driven harmonic oscillator with time-dependent frequency. The second class is related to the existence of additional invariants in the Hamiltonian, which can be used to map the solution of the time-dependent problem to that of a related time-independent one. In this paper we discuss and develop analytic methods for solving time-dependent tunnelling problems, which cannot be addressed by using quadratic Hamiltonians. Specifically, we give an analytic solution to the problem of tunnelling from an attractive time-dependent potential which is embedded in a long-range repulsive potential. Recent progress in atomic physics makes it possible to observe experimentally time-dependent phenomena and record the probability distribution over a long range of time. Of special interest is the observation of macroscopical quantum-tunnelling phenomena in Bose–Einstein condensates with time-dependent trapping potentials. We apply our model to such a case in the last section.

PACS numbers: 03.65.Xp, 31.15.Kb, 03.75.Lm

1. Solutions to time-dependent problems in quantum mechanics

In this paper, we construct the time-evolution operator (or propagator) $K(x, t|x', t')$ for some (time-dependent) potentials in position space. The use of the propagator enables us to discuss the time evolution of an initial state $\psi(x, t')$, which is given at time t' :

$$\psi(x, t) = \int dx' K(x, t|x', t')\psi(x', t'). \quad (1)$$

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Table 1. Known analytic solutions to time-dependent Hamiltonians $H(t)$ that are not quadratic in x, p .

Hamiltonian $H(t)$	Reference
$H(t) = \frac{p^2}{2m} - V_0\delta(x)/t$	[3]
$H(t) = \frac{p^2}{2m} - V_0\delta(x)/\sqrt{t^2 + Bt + C}$	[5]
$H(t) = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 - V_0 e^{-\omega t} \delta(x)$	This work

Surprisingly, it is very hard to find analytic expressions for the propagator (for a comprehensive list see [1]). Time-dependent problems in quantum mechanics are divided into two groups:

- (i) The sudden switch from one Hamiltonian to another at time t' , where both Hamiltonians are not time-dependent:

$$H = \begin{cases} H_- & \text{if } t < t' \\ H_+ & \text{if } t \geq t'. \end{cases} \quad (2)$$

The sudden change of the potential will lead to a spatial time evolution of the wavefunction $\psi(x, t)$, since in general a stationary eigenstate of H_- will not be a stationary eigenstate of H_+ . In this sense, H_- generates an initial wavepacket, which will propagate under the influence of H_+ for $t > t'$.

- (ii) Explicit time dependence due to a continuously changing Hamiltonian $H(t)$ for $t \geq 0$.

In the first case, one can expand the given eigenstate of H_- in terms of the eigenstates of H_+ , or if one models a continuous source of particles (i.e. a particle beam) one can use the powerful method of Laplace transforms, either in an approach that directly incorporates the initial wavefunction $\psi(x, t')$, or by construction of the energy-dependent Green function. Instructive examples are the problem of diffraction in time (the Moshinsky shutter) [2] and the evolution of quantum states in external fields [3, 4].

Unfortunately, the method of Laplace transforms does not cope well with time-dependent potentials. For this type of problem, the Laplace transform gives rise to integro-differential equations. From a conceptual point of view this obstacle is not unexpected, since it is no longer possible to work with a conserved energy. Analytic expressions for propagators are only known for two remaining subclasses:

- (1) quadratic Hamiltonians with time-dependent coefficients (i.e. the driven harmonic oscillator with a time-dependent frequency) [1],
- (2) time-dependent potentials that possess a time-dependent scaling parameter (all the potentials in table 1 belong to this class) [3, 5, 6].

Of special interest for tunnelling phenomena are potentials that possess a dip and a barrier through which an initially confined quantum state can decay (see figure 3). Commonly used models for tunnelling barriers are cubic, Eckart-type or δ potentials [7–10].

2. Tunnelling out of a one-dimensional quantum well

In this work we will derive a novel analytic solution for the tunnelling out of a quantum well (modelled by an attractive δ -potential $V(x, t) = -V(t)\delta(x)$) which is embedded into a repelling oscillator potential $H_\Gamma = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2$. We will discuss and compare three different

cases:

(i) The propagator for an inverted oscillator potential:

$$H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2. \quad (3)$$

(ii) The propagator for a time-independent attractive δ -potential embedded in a repelling oscillator:

$$H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 - V_0\delta(x). \quad (4)$$

(iii) The propagator for an increasingly less attractive δ -potential embedded in a repelling oscillator:

$$H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 - V_0 e^{-\omega t} \delta(x). \quad (5)$$

Since we are interested in the decay rate of the initially confined state, we have to define a quantity that will convey this information. One possibility is to calculate the ‘survival probability’, which is the overlap of an initial state with the state at a later time:

$$P(t) = \left| \int dx \int dx' \psi^*(x, 0) K(x, t|x', 0) \psi(x', 0) \right|^2 = \left| \int dx \psi^*(x, 0) \psi(x, t) \right|^2. \quad (6)$$

For the initial state, we will always choose the bound eigenstate of the Hamiltonian $H_\delta = \frac{p^2}{2m} - V_0\delta(x)$:

$$\psi_\delta(x) = \frac{\sqrt{mV_0}}{\hbar} \exp\left[-\frac{mV_0|x|}{\hbar^2}\right]. \quad (7)$$

2.1. Propagation under the influence of an inverted oscillator

The propagator for

$$H_\cap = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 \quad (8)$$

is given by setting $\omega = i\omega_{\text{at}}$ in the propagator for the attractive oscillator [11, 1]:

$$K_\cap(x, t|x', 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sinh(\omega t)}} \exp\left[\frac{im\omega}{\hbar} \frac{(x^2 + x'^2) \cosh(\omega t) - 2xx'}{2 \sinh(\omega t)}\right]. \quad (9)$$

The time evolution of the initial state $\psi_\delta(x)$ (7) can be expressed in terms of the Moshinsky function $M(x, k, t)$ [2, 12], which may be defined in different ways:

$$M(x, k, t) = \int_{-\infty}^0 dx' \frac{1}{\sqrt{2\pi it}} \exp\left[ikx' + i\frac{(x-x')^2}{2t}\right] \quad (10)$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} dk \frac{\exp\left[ikx - \frac{i}{2}k^2 t\right]}{k - k} \quad (11)$$

$$= \frac{1}{2} \exp\left[ikx - \frac{i}{2}k^2 t\right] \operatorname{erfc}\left[e^{-i\pi/4} \frac{(x-kt)}{\sqrt{2t}}\right]. \quad (12)$$

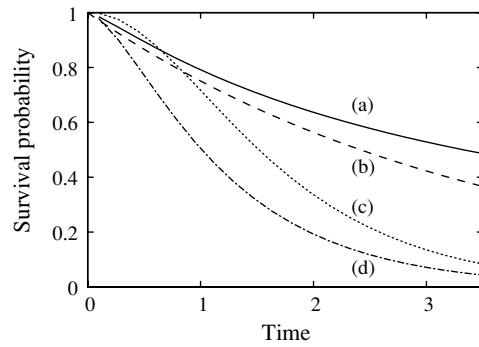


Figure 1. Survival probability of the initial state $\psi_\delta(x) = e^{-|x|}$ for different potentials: (a) free particle, (b) inverted oscillator and static δ -potential, (c) inverted oscillator and exponentially decreasing δ -potential, (d) inverted oscillator potential. Parameters: atomic units ($m = \hbar = 1$), $\omega = 1$, $V_0 = 1$.

Using equations (1) and (10), we obtain the wavefunction at a later time t :

$$\begin{aligned} \psi_\cap(x, t) &= \sqrt{\frac{m^2 \omega V_0}{2\pi i \hbar^3 \sinh(\omega t)}} \int_{-\infty}^{\infty} dx' \exp \left[\frac{i m \omega}{\hbar} \frac{(x^2 + x'^2) \cosh(\omega t) - 2x x'}{2 \sinh(\omega t)} - \frac{m V_0 |x'|}{\hbar^2} \right] \\ &= \frac{1}{\hbar} \sqrt{\frac{m V_0}{\coth(\omega t)}} \left[M \left(x, -\frac{i m V_0}{\hbar^2} + \frac{x m \omega}{\hbar} \left(\frac{1}{\tanh(\omega t)} - \frac{1}{\sinh(\omega t)} \right), \frac{\hbar \tanh(\omega t)}{m \omega} \right) \right. \\ &\quad \left. + M \left(-x, -\frac{i m V_0}{\hbar^2} - \frac{x m \omega}{\hbar} \left(\frac{1}{\tanh(\omega t)} - \frac{1}{\sinh(\omega t)} \right), \frac{\hbar \tanh(\omega t)}{m \omega} \right) \right]. \end{aligned} \quad (13)$$

The corresponding survival probability is shown in figure 1. Not surprisingly, the presence of a repelling potential expedites the decay process compared to the free evolution (with $\omega \rightarrow 0$) of the wave packet, which is also displayed.

2.2. Tunnelling from a time-independent quantum well

A more interesting case is the combination of a repelling long-range potential with an attractive short-range potential. A general method to derive the propagator for the combination of a δ -function potential and another potential is based on the solution of the Lippmann–Schwinger equation by using a Laplace transform (for examples and a discussion see [3], section 3.4). This method is well suited for time-independent Hamiltonians of the form

$$H = H_0 - V_0 \delta(x). \quad (14)$$

In this case the Lippmann–Schwinger equation becomes

$$\begin{aligned} K(x, t|x', 0) &= K_0(x, t|x', 0) \\ &\quad - \frac{i}{\hbar} \int_0^t dt'' \int_{-\infty}^{\infty} dx'' K_0(x, t|x'', t'') V(x'', t'') K(x'', t''|x', 0) \\ &= K_0(x, t|x', 0) + \frac{i V_0}{\hbar} \int_0^t dt'' K_0(x, t|0, t'') K(0, t''|x', 0), \end{aligned} \quad (15)$$

where K_0 denotes the propagator for the simpler Hamiltonian H_0 . In order to solve this equation for the propagator K of the full Hamiltonian H , it is of advantage to switch to

a Laplace transformed picture. The Laplace transform of the propagator is conveniently expressed in terms of the energy-dependent Green function $G(x, x'; E)$ [13]:

$$G(x, x'; E) = \frac{1}{i\hbar} \lim_{\eta \rightarrow 0} \int_0^\infty dt e^{i(E+i\eta)t/\hbar} K(x, t|x', 0). \tag{16}$$

If we apply the Laplace transform to both sides of equation (15), we can use the convolution theorem for a product of two functions (see [14], 29.2.8) and obtain the Green G function of the full Hamiltonian H in terms of the Green function G_0 for H_0 :

$$G(x, x'; E) = G_0(x, x'; E) - V_0 \frac{G_0(x, 0; E)G_0(0, x'; E)}{1 + V_0G_0(0, 0; E)}. \tag{17}$$

The inverse Laplace-transform yields the propagator:

$$K(x, t|x', 0) = K_0(x, t|x', 0) + \frac{V_0}{2\pi i} \int_{-\infty}^\infty dE e^{-iEt/\hbar} \frac{G_0(x, 0; E)G_0(0, x'; E)}{1 + V_0G_0(0, 0; E)}. \tag{18}$$

For the case of a δ -function potential on top of an inverted oscillator we have

$$H = H_0 + V(x), \quad H_0 = \frac{p^2}{2m} - \frac{1}{2}m\omega^2x^2 \quad \text{and} \quad V(x) = -V_0\delta(x). \tag{19}$$

The Laplace transform of the propagator for the inverted oscillator (9) gives rise to the parabolic cylinder function (we use [15], 3.547(10) (there is a misprint, the correct result is $\int_0^\infty \exp\{-2(\alpha \coth x + px)\} \sinh^{2\beta} x dx = \frac{1}{2}\alpha^\beta \Gamma(p - \beta) W_{-p, \beta + \frac{1}{2}}(4\alpha)$) and 9.240):

$$\begin{aligned} G_0(x, 0; E) &= \frac{1}{i\hbar} \int_0^\infty dt e^{iEt/\hbar} K_{\cap}(x, t|0, 0) \\ &= \frac{2^{-3/4+E/(2i\hbar\omega)}}{i\hbar} \sqrt{\frac{m}{i\pi\hbar\omega}} \Gamma\left(\frac{1}{4} + \frac{E}{2i\hbar\omega}\right) D_{-\frac{1}{2} - \frac{E}{i\hbar\omega}}\left(\sqrt{2}|x|\sqrt{\frac{m\omega}{i\hbar}}\right). \end{aligned} \tag{20}$$

At the origin this expressions simplifies to

$$G_0(0, 0; E) = \frac{1}{2i\hbar\omega} \sqrt{\frac{m\omega}{\pi i\hbar}} B\left[\frac{E}{2i\hbar\omega} + \frac{1}{4}, \frac{1}{2}\right], \tag{21}$$

where $B[x, y]$ denotes Euler's beta function (see [14], 6.2). In principle, we could now carry out the inverse Laplace transform by integrating along the real energy axis. But since our main interest is to determine the tunnelling rate for this type of potential, another method is available to extract this information. Following the procedure suggested by Ludviksson for the combination of a δ -potential with a linear force field [16], we locate the poles of the denominator in equation (17) in the complex energy plane. For the inverted oscillator, they are given by the roots of the transcendental equation

$$1 + V_0G_0(0, 0; E) = 0. \tag{22}$$

The determination of the roots (denoted by $E_\delta^{(i)}$) has to be done numerically. Note that there is another set of poles E_Γ present, which results from the Γ -function in $G_0(x, 0; E)$:

$$E_\Gamma^{(k)} = -i\hbar\omega \left(k + \frac{1}{2}\right), \quad k = 0, 2, 4, \dots \tag{23}$$

All poles have a negative imaginary part and thus lead to a decaying wavefunction for $t \rightarrow \infty$ and fixed locations x, x' . However, none of the poles should be included in an evaluation of the inverse Laplace transform, since the original contour runs along the real energy axis and cannot be closed in the lower imaginary E -plane: the residues at the poles with negative imaginary part correspond to wavefunctions that are not bounded at infinity (see also the discussion in [16]). However, the first pole $E_\delta^{(1)}$ lies closest to the real energy axis and is directly connected

to the presence of the δ -potential: for $\omega \rightarrow 0$ the pole is located at $E = -mV_0^2/(2\hbar^2)$, which is precisely the binding energy of the δ -potential. The inverted-oscillator potential leads to a shift of this pole to

$$E_\delta^{(1)} = E_r - \frac{i}{2}E_i, \quad E_r, E_i \in \mathbf{R}. \quad (24)$$

An estimate for the lifetime of the decaying state is thus given by

$$T_{\text{life}} = \frac{\hbar}{E_i}. \quad (25)$$

Figure 1 shows that the exponential decay rate is slowed down compared to the evolution in a purely inverted oscillator potential due to the attraction of the δ -potential.

2.3. Time-dependent δ -potential and an inverted oscillator

Interestingly, it is possible to obtain an exact analytic expression for the propagator of a time-dependent δ -potential on top of an inverted harmonic oscillator. The basic idea is to employ a mapping procedure, which transforms the explicit time-dependent problem into a related time-independent one. Details of this method are given in [1, 5, 6]. The potential must have the following explicit time dependence:

$$V(x, t) = \frac{1}{C^2(t)} V\left(\frac{x}{C(t)} + A(t)\right) + g_1(t)x + g_2(t)x^2 + g_0(t), \quad (26)$$

where the time-dependent coefficients $C(t)$, $A(t)$, $g_0(t)$, $g_1(t)$, $g_2(t)$ are coupled by a set of ordinary differential equations:

$$\ddot{C}(t) - 2g_2(t)C(t) = K/C^3(t), \quad (27)$$

$$\ddot{A}(t) + \frac{2\dot{A}(t)\dot{C}(t)}{C(t)} + \frac{KA(t)}{C^4(t)} - \frac{g_1(t)}{C(t)} = 0, \quad (28)$$

and where K denotes an arbitrary constant. If we choose for the potential the form

$$V(x, t) = -\frac{1}{2}m\omega^2x^2 - e^{-\omega t}V_0\delta(x), \quad (29)$$

we can fulfil all the differential equations by setting

$$C(t) = e^{\omega t}, \quad A(t) = g_0(t) = g_1(t) = K = 0, \quad g_2(t) = -\frac{1}{2}m\omega^2. \quad (30)$$

Following [6], we can now construct an equivalent time-independent Hamilton operator with a potential $\tilde{V}(\tilde{x})$ by transforming the position operator and the time according to

$$\tilde{x} = \frac{x}{C(t)} = e^{-\omega t}x, \quad \tilde{x}' = \frac{x'}{C(0)} = x', \quad \tilde{t} = \int_0^t \frac{ds}{C^2(s)} = \frac{1 - e^{-2\omega t}}{2\omega}. \quad (31)$$

The propagator for the transformed Hamiltonian

$$\tilde{H} = \frac{\tilde{p}^2}{2m} + \tilde{V}(\tilde{x}) = \frac{\tilde{p}^2}{2m} - V_0\delta(\tilde{x}) \quad (32)$$

can be derived with the method of the previous section with the free particle Green function ($k = \sqrt{2mE}/\hbar$)

$$G_{\text{free}}(x, x'; E) = -\frac{im}{\hbar^2 k} e^{ik|x-x'|}, \quad K_{\text{free}}(x, t|x', 0) = \sqrt{\frac{m}{2\pi\hbar t}} \exp\left[\frac{im(x-x')^2}{2\hbar t}\right]. \quad (33)$$

The inverse Laplace transform is conveniently expressed in terms of the Moshinsky function (see, e.g., equation (3.43) in [3], replace V_0 by $-V_0$):

$$\tilde{K}(\tilde{x}, \tilde{t}|\tilde{x}', 0) = K_{\text{free}}(\tilde{x}, \tilde{t}|\tilde{x}', 0) + \frac{mV_0}{2\pi i\hbar^2} \int_{-\infty}^{\infty} dk \frac{\exp\left[\frac{-i\hbar k^2 \tilde{t}}{2m} + ik(|\tilde{x}| + |\tilde{x}'|)\right]}{k - imV_0/\hbar^2} \quad (34)$$

$$= K_{\text{free}}(\tilde{x}, \tilde{t}|\tilde{x}', 0) + \frac{mV_0}{\hbar^2} \text{M}(|\tilde{x}| + |\tilde{x}'|, +imV_0/\hbar^2, \hbar\tilde{t}/m). \quad (35)$$

The mapping is associated with a canonical transformation that gives rise to additional factors in the propagator [6]:

$$K(x, t|x', 0) = \frac{1}{\sqrt{C(t)C(0)}} \exp\left[\frac{im}{2\hbar} \left(x^2 \frac{\dot{C}(t)}{C(t)} - x'^2 \frac{\dot{C}(0)}{C(0)}\right)\right] \tilde{K}(\tilde{x}, \tilde{t}|\tilde{x}', 0) \quad (36)$$

$$= \exp\left[-\frac{\omega t}{2}\right] \exp\left[\frac{im\omega(x^2 - x'^2)}{2\hbar}\right] \tilde{K}\left(e^{-\omega t}x, \frac{1 - e^{-2\omega t}}{2\omega} \middle| x', 0\right). \quad (37)$$

Inserting equation (34) shows that the exponential factor $\exp[-\frac{\omega t}{2}]$ is compensated by $K_{\text{free}}(\tilde{x}, \tilde{t}|\tilde{x}', 0)$, and the free propagator is mapped into the one of the inverted oscillator (as expected from equation (29) in the limit $V_0 \rightarrow 0$):

$$\exp\left[-\frac{\omega t}{2}\right] \exp\left[\frac{im\omega(x^2 - x'^2)}{2\hbar}\right] K_{\text{free}}(\tilde{x}, \tilde{t}|\tilde{x}', 0) = K_{\cap}(x, t|x', 0). \quad (38)$$

Therefore we may rewrite equation (36) like equation (18) as the sum of the propagator for the inverted oscillator K_{\cap} and a contribution due to the time-dependent potential K_V :

$$K(x, t|x', 0) = K_{\cap}(x, t|x', 0) + K_V(x, t|x', 0), \quad (39)$$

with

$$K_V(x, t|x', 0) = \frac{mV_0}{\hbar^2} e^{im\omega(x^2 - x'^2)/(2\hbar) - \frac{\omega t}{2}} \text{M}\left(e^{-\omega t}|x| + |x'|, \frac{imV_0}{\hbar^2}, \frac{\hbar(1 - e^{-2\omega t})}{2\omega m}\right). \quad (40)$$

The corresponding survival probability of the initial state (7) is displayed in figure 1. As expected, the decreasing δ -potential strength expedites the decay process. The asymptotic behaviour for $\omega t \gg 1$ can be extracted by inserting $\lim_{\omega t \rightarrow \infty} \tilde{x} = 0$, and $\lim_{\omega t \rightarrow \infty} \tilde{t} = 1/(2\omega)$ into equation (34):

$$K(x, t|x', 0) \approx \exp[-\omega t/2] [e^{+im\omega x^2/(2\hbar)}] [e^{-im\omega x'^2/(2\hbar)} \tilde{K}(0, 1/(2\omega)|x', 0)]. \quad (41)$$

Using equation (6) we obtain an exponentially decaying survival probability

$$P(t) \approx e^{-\omega t} \left| \int dx e^{im\omega x^2/(2\hbar)} \psi^*(x, 0) \right|^2 \left| \int dx' e^{-im\omega x'^2/(2\hbar)} \tilde{K}\left(0, \frac{1}{2\omega} \middle| x', 0\right) \psi(x', 0) \right|^2. \quad (42)$$

Note that this behaviour is independent of the choice of $V(x)$ in equation (26).

3. The optical atom laser as a time-dependent problem

As an application of our method we discuss the time evolution of the quantum-state of a Bose–Einstein condensate (BEC) that is confined in a time-dependent trapping potential. Recent experiments use optically trapped BECs of rubidium atoms to produce a matter wave of macroscopic dimensions [17]. Optical traps utilize the polarizability of atoms to achieve a

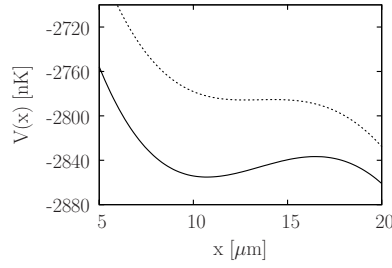


Figure 2. Optical trapping potential for rubidium atoms in the presence of the gravitational force along the x -axis: $V(x) = -V_0 \exp(-2x^2/w_0^2) - Fx$. Parameter: lower curve $V_0 = 2400$ nK, upper one: $V_0 = 2286$ nK, $w_0 = 27$ μm , $F = 103$ nK μm^{-1} . Only the lower curve can support a metastable state.

BEC in the presence of a running optical wave, which is generated by a focused laser beam. To a good approximation, the effective optical potential of the laser is given by [18]

$$V_{\text{trap}}(x, y, z, t) = -\frac{V(t)}{1 + (z/z_0)^2} \exp\left[-\frac{2(x^2 + y^2)}{w_0^2} \frac{1}{1 + (z/z_0)^2}\right], \quad (43)$$

where

$$z_0 = \frac{\pi w_0^2}{\lambda_{\text{laser}}}, \quad \text{and} \quad V(t) = \frac{\alpha_s P_{\text{laser}}(t)}{\pi \epsilon_0 c w_0^2}. \quad (44)$$

Typical experimental parameters are reported in [17]: the laser is characterized by its minimal beam waist w_0 (27 μm), its wavelength λ_{laser} (10.6 μm) and an adjustable power P_{laser} (200–0 mW). The polarizability α_s of ^{87}Rb atoms is about 5.3×10^{-39} m² C V⁻¹. The other constants denote the permittivity of the vacuum ϵ_0 and the speed of light c .

Since the laser is operated in the presence of the gravitational field $F\hat{e}_x$ of the earth, the Hamiltonian for atoms trapped in the laser field becomes

$$H(t) = \frac{\mathbf{p}^2}{2m} - \frac{V(t)}{1 + (z/z_0)^2} \exp\left[-\frac{2(x^2 + y^2)}{w_0^2} \frac{1}{1 + (z/z_0)^2}\right] - Fx, \quad (45)$$

where $F = mg$, ($g \approx 9.81$ m s⁻², $m \approx 87$ atomic mass units for ^{87}Rb). The depth of the optical potential $V(t)$ is freely adjustable and can be put as a function of time. If one reduces the laser power slowly, one can achieve a beam of atoms that leave the trapping potentials once it becomes unstable due to the gravitational force. Figure 2 shows a cut trough the potential along the x -direction with $y = z = 0$ for two different laser powers. The optical potential forms a quantum well, albeit in three dimensions, where the repulsive quadratic potential in (5) is replaced by a linear potential.

In order to model the gradually switching off of the optical trapping potential, one should take into account the asymmetry of the potential due to the gravitational force. We would also like to use the methods developed in section 2.3. If we choose the potential of the form

$$V_{\text{ch}}(x) = -\frac{\alpha\mu^2\hbar^2}{m\text{ch}^2(\alpha, \mu, x)}, \quad \text{with} \quad \text{ch}(\alpha, \mu, x) = \frac{1}{2}(\alpha e^{\mu x} + e^{-\mu x}), \quad (46)$$

we can use the known propagator [19]

$$\begin{aligned} K_{\text{ch}}(x, x'; t) &= K_{\text{free}}(x, x', t) + \frac{\alpha\mu}{2\text{ch}(\alpha, \mu, x)\text{ch}(\alpha, \mu, x')} \\ &\times [e^{i\mu^2\hbar t/(2m)} - (e^{\mu(x'-x)}\mathbf{M}(x-x', -i\mu, t\hbar/m) \\ &+ e^{\mu(x-x')}\mathbf{M}(x'-x, -i\mu, t\hbar/m))] \end{aligned} \quad (47)$$

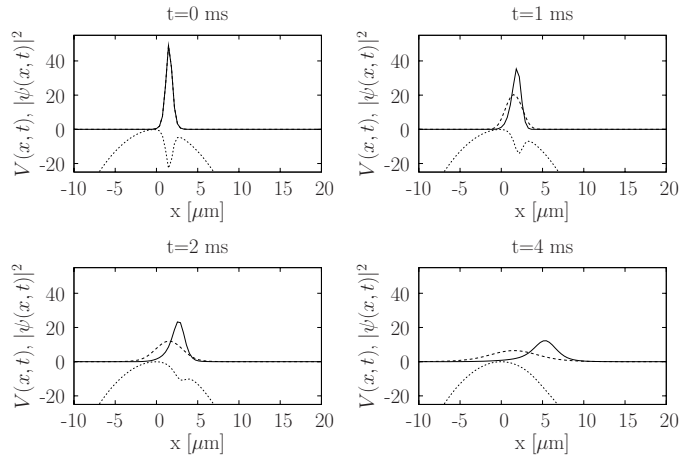


Figure 3. Time evolution of an initially confined wavepacket in a time-dependent potential. The dotted line shows the potential given in equation (48), the dashed line shows the force-free time evolution of the initial state, and the solid line denotes $|\psi(x, t)|^2$ from equation (50). Parameter: $\alpha = 1/450$, $\mu = 2 \times 10^6/\text{m}$, $\omega = 2\pi \times 50 \text{ Hz}$, the potential energy scale is nK, and arbitrary units. for $|\psi(x, t)|^2$.

to model a time-dependent potential similar to the one used in equation (29)

$$V_{\text{ch}}(x, t) = -\frac{1}{2}m\omega^2x^2 + \frac{V_{\text{ch}}(x/C(t))}{C^2(t)}, \quad \text{with } C(t) = e^{\omega t}. \quad (48)$$

A realistic (albeit one-dimensional) picture of the time evolution of this potential is given in figure 3. Using the same transformation as before, we can derive the time evolution of an initial state, which we choose in this case to be the bound eigenstate in the potential $V_{\text{ch}}(x)$:

$$\psi_{\text{ch}}(x, t=0) = \frac{\sqrt{2\alpha\mu}}{2\text{ch}(\alpha, \mu, x)}. \quad (49)$$

The wavefunction at a later time becomes

$$\psi(x, t) = e^{-\omega t/2} \int_{-\infty}^{\infty} dx' e^{im\omega(x^2-x'^2)/(2\hbar)} K_{\text{ch}}\left(e^{-\omega t}x, \frac{1-e^{-2\omega t}}{2\omega} \middle| x', 0\right) \psi_{\text{ch}}(x', 0). \quad (50)$$

The integration has to be carried out numerically. Figure 3 shows the resulting time evolution of the probability density for different times. The presence of the potential well delays the broadening of the initial wavepacket compared to a force-free evolution (also shown in the figure). This behaviour of the model is not desirable if one wishes to model wavepackets of longer spatial extension. The experiments so far tried to create an elongated wavepacket. To achieve this goal experimentally requires a careful tuning of the time dependence of the trapping potential [20]. Our simple one-dimensional model shows the coherent control of the initial wavepacket by the changing potential and may describe a more focused atom laser.

A one-dimensional treatment as presented here should reproduce the main features of the experiment, since the tunnelling in the directions perpendicular to the gravitational field are not important, because the lowering of the potential is related mainly to the gravitational potential $-Fx$. For an accurate theory of the experimentally realized atom laser [17], one should employ a three-dimensional trapping potential and a deeper optical potential, which can accommodate several bound states. A detailed model of an optical trap by reflectionless

potentials [21] is another possibility of obtaining potentials which still admit analytic solutions.

4. Conclusions

We have conducted a systematic study of tunnelling phenomena from an attractive potential embedded into an inverted oscillator barrier. The use of the propagator makes it possible to actually obtain the complete time evolution of the wavefunction analytically for an exponentially decaying quantum well. Our results are relevant for the study of tunnelling processes in time-dependent confining potentials, such as the trapping potentials for condensed bosons (BEC). We succeeded in finding new analytic results for one-dimensional tunnelling processes through a time-dependent barrier. The analytic results provide important reference cases for other methods, such as the semiclassical WKB approximation [7, 8]. The mapping procedure of section 2.3, in principle, is not limited to one-dimensional problems and opens the way to the study of the propagation of wavepackets through explicit time-dependent barriers. The experimental realization of coherent atomic ensembles in controllable confining potentials demonstrates the need for theoretical descriptions of tunnelling processes through time-dependent barriers.

Acknowledgments

We appreciate stimulating discussions with M Kleber, P Kramer, V Man'ko, M Weitz, and financial support by CONACyT. TK would like to thank the members of the Instituto de Física for their hospitality during the completion of this work.

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